Rotating cosmic strings and gravitational soliton waves

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Using the Belinsky-Zakharov inverse scattering method, a four-parameter family of cylindrically symmetric solutions of Einstein's vacuum equations is obtained. They are of Petrov type D and, for a certain range of the parameters, represent solitonlike gravitational waves interacting with a straight-line cosmic string. Their interpretation as rotating cosmic strings occupying the axis of symmetry is also investigated.

I. INTRODUCTION

Cylindrically symmetric solutions of the Einstein vacuum equations have attracted considerable interest recently. The following are probably the main stimuli of this development.

First, spacetime models with cylindrical symmetry are pertinent to the cosmic strings predicted to form during phase transitions in the early Universe by current grand unified theories[1] (GUT's). Time-independent, flat models of this kind, having a conical singularity centered on the symmetry axis, have been associated with static straight-line cosmic strings and effects such as multiple-image production (lensing) and discontinuities of the microwave background have been predicted.[2–7] Consequently, time-dependent solutions with cylindrical symmetry can model the interaction of cosmic strings with gravitational waves, for example.

The second reason for the recent increase of interest in the above type of solutions is endogenous to general relativity, in the following sense. The complexity of the corresponding field equations in the time-dependent case is such that exact solutions have been very difficult to obtain. The well-known Einstein-Rosen[8] waves give an example, but one in which only one of the two dynamical degrees of freedom that are possible is present. This situation, however, changed drastically in the last few years thanks to the soliton technique of solving Einstein's equations developed by Belinsky and Zakharov[9,10] and the results in the same direction coming from investigations by Chandrasekhar, Ferrari, and Xanthopoulos.[11–14] The method developed by the latter is based on the concept of the Ernst potential and the analogies existing between plane waves and solutions with cylindrical symmetry. Following this approach, Xanthopoulos[13,14] obtained a solution representing a rotating cosmic string surrounded by cylindrical gravitational waves which he later extended so as to include electromagnetic waves, too.[15]

The Belinsky-Zakharov approach, on the other hand, forms the basis of the Garriga-Verdaguer[16] solution, which includes a class of metrics representing straight-line cosmic-string interacting with solitonlike Einstein-Rosen gravitational waves. This most interesting solution, being radiating, has led its authors to conclude that it is qualitatively different from the one found by Xanthopoulos, which, according to Xanthopoulos himself, is nonradiating.

The solution presented in this paper is also a product of the application of the Belinsky-Zakharov method of integrating Einstein's equations when the corresponding metric is characterized by cylindrical symmetry. Specifically, it is obtained as the two-soliton product which results by using as "seed" Minkowski's metric in cylindrical coordinates. In fact, it is a four-parameter family of radiating solutions which, when the parameters are chosen appropriately, represents solitonlike gravitational waves interacting with a straight-line cosmic string.

It turns out that the present solution is a one-parameter generalization of the Xanthopoulos solution mentioned above, with which it shares all of the basic physical features. This implies that the latter solution is also radiating and the conclusions of its author regarding this point must be corrected. It also implies that the claim for a qualitative distinction between the solution of Xanthopoulos and the one by Garriga and Verdaguer mentioned above cannot be supported. In fact, the Xanthopoulos solution, as well the one presented in this paper, are more general than the one given by Garriga and Verdaguer, in the sense that it includes both radiation modes that are possible in cylindrical gravitational waves, the $+$ as well as the $\times$ one, while in the latter solution only the $+$ mode is present.

The present solution is of Petrov type D. Therefore, it must be a member of the Kinnersley[17] family of solutions, but we have not tried to show this explicitly, the emphasis of this paper being on the method of derivation as well as on the physical interpretation of the solution. Thus, we devote Sec. II to outlining the Belinsky-Zakharov soliton technique and Sec. III to the application of this method to cylindrically symmetric metrics obtained by taking Minkowski's metric as a seed. In Sec. IV we construct the particular class of metrics that represent solitonlike gravity waves propagating in free space or in the environment of straight-line cosmic strings. Finally, in Sec. V, we study the asymptotic behavior of the metric, the C energy and Weyl tensor corresponding to the solution presented in the previous section and put forward the physical interpretation that can be supported on the basis of the above analysis. Throughout the paper we use units in which the speed of light in vacuum and the gravitational constant equal unity.

II. OUTLINE OF THE BELINSKY-ZAKHAROV METHOD

Consider the cylindrically symmetric line element

$$ds^2=f(\rho,t)(d\rho^2-dt^2)+g_{ab}(\rho,t)dx^a dx^b\ , \tag{2.1}$$

where the indices a,b run from 1 to 2 and

$$x^1=z,\quad x^2=\phi\ . \tag{2.2}$$

The range of the coordinates (ρ,z,ϕ) is the usual one for the cylindrical polar system in Euclidean space.

The vacuum Einstein equations for the above line element can be written as[9]

$$A_{,\eta}=B_{,\zeta}\ , \tag{2.3}$$

$$(\ln f)_{,\zeta}=\frac{(\ln\alpha)_{,\zeta\zeta}}{(\ln\alpha)_{,\zeta}}+\frac{\mathrm{tr}A^2}{4\alpha\alpha_{,\zeta}}\ , \tag{2.4}$$

$$(\ln f)_{,\eta}=\frac{(\ln\alpha)_{,\eta\eta}}{(\ln\alpha)_{,\eta}}+\frac{\mathrm{tr}B^2}{4\alpha\alpha_{,\eta}}\ , \tag{2.5}$$

where

$$A\equiv-\alpha g_{,\zeta}g^{-1},\quad B\equiv\alpha g_{,\eta}g^{-1}\ , \tag{2.6}$$

$$g\equiv(g_{ab}),\quad \alpha\equiv[\det(g_{ab})]^{1/2}\ , \tag{2.7}$$

(η,ζ) are null coordinates defined by the relations

$$t=\zeta-\eta,\quad \rho=\zeta+\eta\ , \tag{2.8}$$

and $(\ )_{,x}\equiv\partial(\ )/\partial x$.

Equation (2.3) is the integrability condition of Eqs. (2.4) and (2.5) and its trace reads

$$\alpha_{,\zeta\eta}=0\ . \tag{2.9}$$

Therefore,

$$\alpha=a(\zeta)+b(\eta)\ , \tag{2.10}$$

where a,b are arbitrary functions of the indicated arguments. A second solution of Eq. (2.9) is given by

$$\beta=a(\zeta)-b(\eta) \tag{2.11}$$

and this is independent of α, provided $ab\neq0$.

In the Belinsky-Zakharov (BZ) inverse scattering method (ISM) Eq. (2.3) is replaced by the "Schrödinger equations"

$$D_\zeta\psi=\frac{A}{\lambda-\alpha}\psi\ , \tag{2.12a}$$

$$D_\eta\psi=\frac{B}{\lambda+\alpha}\psi \tag{2.12b}$$

for the 2×2 matrix "wave function" $\psi(\zeta,\eta;\lambda)$. Here, λ is a complex "spectral parameter," and

$$D_\zeta\equiv\partial_\zeta-\frac{2\lambda}{\lambda-\alpha}\partial_\lambda\ , \tag{2.13a}$$

$$D_\eta\equiv\partial_\eta+\frac{2\lambda}{\lambda-\alpha}\partial_\lambda\ . \tag{2.13b}$$

The relation between the wave function ψ and the matrix g is given by the equation

$$g=\psi\,|_{\lambda=0}\equiv\psi(\eta,\zeta;0)\ . \tag{2.14}$$

Actually, BZ have shown that, given a solution $\psi^{(0)}$ of Eq. (2.12), one can construct a new solution of the same equation by a series of algebraic operations which start with the ansatz

$$\psi=\chi\psi^{(0)}\ . \tag{2.15}$$

Assuming that the "scattering matrix" χ has N simple poles in the complex λ plane, one finds that their trajectories are given by

$$\mu_k=(w_k-\beta)\pm[(w_k-\beta)^2-\alpha^2]^{1/2}\ , \tag{2.16}$$

where w_k arbitrary complex constants, and that the N-soliton metric (f,g) which is produced given the "seed" $(f^{(0)},g^{(0)})$ is given by

$$f=f^{(0)}\alpha^{-N^2/2}\frac{\left|\prod_{k=1}^{N}\mu_k\right|^{N+1}}{\left|\prod_{\substack{k,l=1\\k>l}}^{N}(\mu_k-\mu_l)^2\right|}\det(\Gamma_{kl})\ , \tag{2.17a}$$

$$g_{kl}=\left|\prod_{k=1}^{N}\frac{\mu_k}{\alpha}\right|g_{kl}^{N}\ , \tag{2.17b}$$

where

$$g_{ab}^{N}=g_{ab}^{(0)}-\sum_{k,l=1}^{N}\frac{(\Gamma^{-1})_{kl}N_a^{(k)}N_b^{(l)}}{\mu_k\mu_l}\ , \tag{2.18a}$$

$$\Gamma_{kl}\equiv\frac{m_a^{(k)}g_{ab}^{(0)}m_b^{(l)}}{\mu_k\mu_l-\alpha^2}\ , \tag{2.18b}$$

$$N_a^{(k)}\equiv m_b^{(k)}g_{ba}^{(0)}\ , \tag{2.18c}$$

$$m_a^{(k)}\equiv m_{0b}^{(k)}M_{ba}^{(k)}\ , \tag{2.18d}$$

$$M_{ab}^{(k)}\equiv[\psi^{(0)}(\eta,\zeta;\mu_k)]_{ab}^{-1} \tag{2.18e}$$

with $m_{0b}^{(k)}$ arbitrary constants.

III. CYLINDRICALLY SYMMETRIC SOLITON SOLUTIONS DERIVABLE FROM MINKOWSKI'S METRIC

When the seed metric is diagonal the solution of Eq. (2.12) for $\psi^{(0)}$ as well as the final formulas given by Eqs. (2.17) and (2.18) simplify considerably.[18–20] Specifically, one can show[20] that the latter can be combined in the following form, which involves determinants of $N\times N$ matrices:

$$f=f^{(0)}\alpha^{-N^2/2}\det(Z_{ij}^{(0)})\frac{\left|\sum_{k=1}^{N}\sigma_k\right|^{N}}{\left[\prod_{k=1}^{N}S_k\right]\left[\prod_{\substack{k,l=1\\k>l}}^{N}(\sigma_k-\sigma_l)^2\right]}\ , \tag{3.1a}$$

$$g_{11}=g_{11}^{(0)}\left|\prod_{k=1}^{N}\sigma_k\right|\frac{\det(Z_{ij}^{(-1)})}{\det(Z_{ij}^{(0)})}\ , \tag{3.1b}$$

$$g_{12}=\alpha\left|\prod_{k=1}^{N}\sigma_k\right|\left[1-\frac{\det(Z_{ij}^{(0)}+S_j/\sigma_i\sigma_j)}{\det(Z_{ij}^{(0)})}\right]\ , \tag{3.1c}$$

$$g_{22}=g_{22}^{(0)}\left|\prod_{k=1}^{N}\frac{1}{\sigma_k}\right|\frac{\det(Z_{ij}^{(1)})}{\det(Z_{ij}^{(0)})}\ , \tag{3.1d}$$

where

$$Z_{ij}^{(\delta)}\equiv\frac{1+S_iS_j(\sigma_i\sigma_j)^{\delta}}{\sigma_i\sigma_j-1} \tag{3.1e}$$

with $\delta=-1,0,1,$

$$\sigma_i\equiv\mu_i/\alpha\ , \tag{3.1f}$$

$$S_i\equiv\frac{Q_i\sigma_i g_{11}^{(0)}}{[\psi_{11}^{(0)}(\zeta,\eta;\mu_i)]^2}\ , \tag{3.1g}$$

and Q_i are arbitrary complex constants, related to the ones appearing in Eq. (2.18) by

$$Q_k=2\frac{w_k m_{01}^{(k)}}{m_{02}^{(k)}}\ . \tag{3.2}$$

Let us consider, as an example, the line element

$$ds_{(0)}^2=-dt^2+d\rho^2+\rho^2d\phi^2+dz^2\ , \tag{3.3}$$

which represents the spacetime of Minkowski in cylindrical coordinates. Comparing it with the line element (2.1) we conclude that

$$f^{(0)}=1,\quad g^{(0)}=\mathrm{diag}(1,\rho^2),\quad \alpha^{(0)}=\rho\ . \tag{3.4}$$

According to Eqs. (2.6) and (2.8), we then have

$$A^{(0)}=-B^{(0)}=\mathrm{diag}(0,-2)\ . \tag{3.5}$$

It is, now, easily verified that

$$\psi^{(0)}=\mathrm{diag}(1,\rho^2+2t\lambda+\lambda^2) \tag{3.6}$$

gives a solution of Eq. (2.12).

Let us, next, choose the "gauge" in which

$$\alpha=\rho,\quad \beta=t\ . \tag{3.7}$$

Then, Eqs. (2.16), (3.1f), and (3.1g) give

$$\mu_k=(w_k-t)\pm[(w_k-t)^2-\rho^2]^{1/2} \tag{3.8}$$

and

$$S_k=Q_k\mu_k/\rho=Q_k\sigma_k\ . \tag{3.9}$$

The last equation, combined with Eqs. (3.1a)–(3.1e), gives the totality of soliton solutions of Einstein's vacuum equations which can be derived by taking Minkowski's metric in cylindrical coordinates.

Consider, in particular, the two-soliton solution belonging to the above class. According to Eqs. (3.1a)–(3.1e), the metric coefficients corresponding to this solution are given by

$$f=c^2Z/|\sigma_1\sigma_2|(\sigma_1^2-1)(\sigma_2^2-1)\ , \tag{3.10a}$$

$$g_{11}=(1/Z)|\sigma_1\sigma_2|[(\sigma_1-\sigma_2)^2(1+Q_1Q_2)^2+(\sigma_1\sigma_2-1)^2(Q_1-Q_2)^2]\ , \tag{3.10b}$$

$$g_{12}=(q/Z)(w_2-w_1)|\sigma_1\sigma_2|\times[(Q_1Q_2+1)(Q_1\sigma_1^2-Q_2\sigma_2^2)-(Q_1Q_2\sigma_1^2\sigma_2^2+1)(Q_1-Q_2)^2]\ , \tag{3.10c}$$

$$g_{22}=\rho^2(1/Z|\sigma_1\sigma_2|)[(\sigma_1-\sigma_2)^2(1+Q_1Q_2\sigma_1^2\sigma_2^2)^2+(\sigma_1\sigma_2-1)^2(Q_1\sigma_1^2-Q_2\sigma_2^2)]\ , \tag{3.10d}$$

$$Z=(\sigma_1-\sigma_2)^2(1+Q_1Q_2\sigma_1\sigma_2)^2+(\sigma_1\sigma_2-1)^2(Q_1\sigma_1-Q_2\sigma_2)^2\ , \tag{3.10e}$$

where c is an arbitrary real constant.

The metric which will be the object of our investigation in the following, is the one that results from Eq. (3.10) by the choice of a plus sign in Eq. (3.8) and

$$w_1=\bar{w}_2=i,\quad Q_1=\bar{Q}_2\equiv Q\ , \tag{3.11}$$

where the overbar denotes complex conjugation and i is the imaginary unit. Introducing the functions (x,y) via the equations

$$t=xy\ , \tag{3.11a}$$

$$\rho=[(x^2+1)(y^2-1)]^{1/2} \tag{3.11b}$$

we can write the corresponding line element in the form

$$ds^2=\frac{c^2X}{x^2+y^2}(-dt^2+d\rho^2)+\frac{X}{Y}\rho^2d\phi^2+\frac{Y}{X}(dz-\omega\,d\phi)^2\ , \tag{3.12}$$

where

$$X\equiv4(Q_Ix+Q_R)^2+[(|Q|^2+1)y-|Q|^2+1]^2\ , \tag{3.13a}$$

$$Y\equiv4Q_I^2(x^2+1)+(|Q|^2+1)^2(y^2-1)\ , \tag{3.13b}$$

$$\omega\equiv(4/Y)\{Q_I(|Q|^2-1)y(x^2+1)+(|Q|^2+1)[Q_Rx(y^2-1)-Q_I(x^2+y^2)]\}\ , \tag{3.13c}$$

IV. GRAVITATIONAL SOLITON WAVES AND COSMIC STRINGS

In order for the solution of Einstein's vacuum equations given by Eqs. (3.12) and (3.13) to represent gravitational waves propagating in empty space or gravitational waves coupled with cosmic strings, the quantity $(g_{\phi\phi}/\rho^2)$ must be independent of ρ in the limit $\rho\to0$. Equations (3.11) and (3.13) show, however, that, for $\rho\approx0$,

$$X\simeq4[1+(Q_It+Q_R)^2]+O(\rho)\ , \tag{4.1a}$$

$$Y\approx4Q_I^2(t^2+1)+O(\rho)\ , \tag{4.1b}$$

$$\omega\approx-(2/Q_I)+O(\rho)\ . \tag{4.1c}$$

These expressions imply that $g_{\phi\phi}=(X/Y)\rho^2+(Y/X)\omega^2$ will not have the required behavior as the axis of symmetry is approached.

In order to remedy this defect, we take advantage of the invariance of the formalism presented in the previous section under linear transformations of the coordinates x^a and replace the original pair (z,ϕ) by (z',ϕ'), where

$$z'=z+(2/Q_I)\phi, \quad \phi'=\phi \ . \tag{4.2}$$

In the next section, it will be shown explicitly that the metric which results from the above transformation contains at most a conical singularity.

Furthermore, in order to bring our results in contact with those of previous investigators, we introduce the real parameters p, q, and l via the equations

$$\frac{1}{q}=-\frac{2Q_R}{|Q|^2+1}, \quad \frac{p}{q}=\frac{2Q_I}{|Q|^2+1}, \quad \frac{l}{q}=\frac{|Q|^2-1}{|Q|^2+1} , \tag{4.3}$$

which imply that

$$q^2-p^2-l^2=1 \ . \tag{4.4}$$

Substituting Eqs. (4.2) and (4.3) in Eqs. (3.13), and dropping the primes, for convenience, we obtain

$$X=(1-px)^2+(l-qy)^2 , \tag{4.5a}$$

$$Y=p^2(x^2+1)+q^2(y^2-1) , \tag{4.5b}$$

$$\omega=(2/pY)[lp^2(x^2+1)(y-1) +q(1+l^2-lq-px)(y^2-1)] \ . \tag{4.5c}$$

Comparison of the above expressions with those given by Eqs. (4.4) and (4.7) of Ref. 14, taking into account that our (x,y,ω) correspond to (η,μ,q_2) of the above Ref. 14, yields the following conclusion. The class of solutions of the Einstein vacuum equations given by Eqs. (3.12), (4.4), and (4.5) is a generalization of the one presented by Xanthopoulos recently. The latter can be obtained from the one presented above by setting the parameter l equal to zero. Actually, the relation between these two classes of solutions is exactly the same as that between the Kerr-Newman-Unti-Tamburino (KNUT) and Kerr families of stationary axially symmetric ones. This is a consequence of the fact that the following series of operations yields the KNUT metric in the (t,r,θ,ϕ) coordinate system of Boyer and Lindquist.[21]

(1) The transformation $z\to z+2(l/p)\phi$, $\phi\to\phi$ turns $\omega(x,y)$ given by Eqs. (4.5c) into

$$\omega'=(2/pY)[lp^2(x^2+1)y+q(1+l^2-px)(y^2-1)]$$

without affecting the rest of the metric coefficients. (ii) Introduce the coordinate (r,θ) and the parameters (m,a,b) via the equations

$$x=(r-m)(a^2-m^2-b^2)^{-1/2}, \quad y=\cosh\theta, \quad c=m \ ,$$

$$p=-(a^2-m^2-b^2)^{1/2}/m, \quad q=a/m, \quad l=b/m \ .$$

(iii) Finally, let

$$\theta\to i\theta, \quad z\to t \ ,$$

$$\phi\to(a^2-b^2-m^2)^{1/2}\phi \ .$$

V. PHYSICAL INTERPRETATION

In order to specify the physical features of the spacetime manifold with the line element given by Eqs. (3.11), (3.12), and (4.5), we turn to an analysis of the behavior shown by its metric, Thorne's C energy, and the Weyl tensor in the following three characteristic regions. Near the axis, at spacelike infinity and the vicinity of the null cone $\rho=|t|$.

A. The metric

The symmetry axis region $\rho \ll |t|$. Here the functions appearing in the line element (3.12) take the following forms:

$$f\approx c^2F(t)\equiv c^2\frac{(l-q)^2+(1-pt)^2}{1+t^2} , \tag{5.1a}$$

$$X\approx[(l-q)^2+(1-pt)^2] \times\left[1+\frac{q(q-l)-pt(pt-1)}{(l-q)^2+(1-pt)^2}\rho^2\right] , \tag{5.1b}$$

$$Y\approx p^2(1+t^2)\left[1+\frac{q^2-p^2t^2}{p^2(1+t^2)^2}\rho^2\right] , \tag{5.1c}$$

$$\omega\approx-\frac{2q(1-lq+l^2-pt)+lp^2(1+t^2)}{p^3(1+t^2)^2}\rho^2 \ . \tag{5.1d}$$

On the basis of these expressions we can write

$$ds^2\approx c^2F\left[-dt^2+d\rho^2+\frac{\rho^2}{c^2p^2}d\phi^2\right]+\frac{p^2}{F}dz^2 \tag{5.2}$$

for the line element in the region near the axis. Equation (5.2) implies that, given a small circle lying in the hypersurface $dt=dz=0$ and having its center at $\rho=0$, the ratio (circumference/radius) differs from 2π, unless $|cp|=1$. Equivalently, when $|cp|\neq 1$ the axis region is characterized by an angle deficit or surplus depending on whether $D_<$, where

$$D_<=2\pi(1-|cp|^{-1}) \tag{5.3}$$

is positive or negative, respectively.

Spacelike infinity $\rho \gg |t|$. In this region

$$f\approx c^2\left[q^2-\frac{2lq}{\rho}\right] , \tag{5.4a}$$

$$X\approx\left[q^2-\frac{2lq}{\rho}\right]\rho^2 , \tag{5.4b}$$

$$Y\approx(q^2\rho^2+p^2-q^2t^2) , \tag{5.4c}$$

$$\omega\approx 2\frac{1+l^2-lq}{pq} \ . \tag{5.4d}$$

Therefore, as one approaches the region where $|t| \ll \rho\to\infty$, the line element takes the form

$$ds^2 \approx c^2q^2 \left[-dt^2+d\rho^2+\frac{\rho^2}{c^2q^2}d\phi^2\right] + \left[dz-\frac{2(1+l^2-lq)}{pq}d\phi\right]^2 . \tag{5.5}$$

The transformation

$$z \rightarrow z - \frac{2(1+l^2-lq)}{pq}\phi \tag{5.6}$$

turns Eq. (5.5) into

$$ds^2 \approx c^2q^2 \left[-dt^2+d\rho^2+\frac{\rho^2}{c^2q^2}d\phi^2\right]+dz^2 . \tag{5.7}$$

It is now obvious that the spacetime under consideration is asymptotically flat at spacelike infinity with an angle deficit or surplus given by

$$D_> = 2\pi(1-|cq|^{-1}) . \tag{5.8}$$

Null infinity $\rho=|t|\rightarrow\infty$. If we move far away from the symmetry axis and consider the region of the null direction $\rho=\pm t$, we find that the quantities given by Eq. (4.5) tend to the following expressions:

$$f \approx \frac{c^2(p^2+q^2)}{2}\left[1-2\frac{\pm p+lq}{p^2+q^2}\frac{1}{\sqrt{\rho}}\right] , \tag{5.9a}$$

$$X \approx (p^2+q^2)\left[\rho-2\frac{\pm p+lq}{p^2+q^2}\sqrt{\rho}\right] , \tag{5.9b}$$

$$Y \approx (p^2+q^2)\left[\rho-\frac{1+l^2}{2(p^2+q^2)}\right] , \tag{5.9c}$$

$$(\omega/2) \approx \frac{(1+l^2-lq)q-lp^2}{p(p^2+q^2)}+\frac{lp\mp q}{p^2+q^2}\sqrt{\rho} . \tag{5.9d}$$

Comparing Eqs. (5.4) and (5.9), we conclude that the rate at which the metric coefficients tend to the ones corresponding to a flat spacetime is slower as we approach infinity along a null direction than a spacelike one. This indicates that our solution represents a gravitational disturbance propagating along the null cone $|t|=\rho$. The results of the following subsections show that this is the case, indeed.

B. The C energy

The quantity

$$C(\rho,t) \equiv \ln|fg_{zz}|^{1/2} \tag{5.10}$$

was introduced by Thorne[22] as a measure of the energy density of cylindrically symmetric gravitational systems and is referred to as their C energy. Specifically, $C(\rho,t)$ is proportional to the total energy per unit z length contained within a cylindrical shell of coordinate radius ρ, while $C_{,\eta}=C_{,\rho}-C_{,t}$ measures the rate at which this energy crosses the shell and is radiated away.

From Eqs. (3.12) and (4.5b) we find that, in our case,

$$2C = \ln\frac{c^2p^2(x^2+1)+c^2q^2(y^2-1)}{x^2+y^2} , \tag{5.11}$$

while

$$C_{,\substack{\eta\\ \zeta}} = \frac{(1+l^2)(x^2+1)^{1/2}(y^2-1)^{1/2}[x(y^2-1)^{1/2}\pm y(x^2+1)^{1/2}]^2}{(x^2+y^2)^2Y} . \tag{5.12}$$

Using the above expressions, one easily finds the asymptotic behavior of the C energy in the three characteristic regions of spacetime described above to be as follows.

(1) For $\rho \ll |t|$,

$$2C \approx \ln\left[c^2p^2+\frac{c^2(1+l^2)\rho^2}{(1+t^2)^2}\right] , \tag{5.13}$$

$$C_{,\eta} \approx \frac{1+l^2}{p^2(1+t^2)^2}\left[\rho+\frac{2t\rho^2}{1+t^2}\right] , \tag{5.14a}$$

$$C_{,\zeta} \approx \frac{1+l^2}{p^2(1+t^2)^2}\left[\rho-\frac{2t\rho^2}{1+t^2}\right] . \tag{5.14b}$$

These expressions show clearly that, in the immediate vicinity of the axis ($\rho\rightarrow 0$), not near the light cone $\rho=|t|$, as well as at timelike infinity ($|t|\rightarrow\infty$), there is no energy flux and the C energy itself takes the constant value

$$C_< = \ln|cp| . \tag{5.15}$$

We can now write Eq. (5.3) as

$$D_< = 2\pi[1-\exp(-C_<)] \tag{5.16}$$

and conclude that $D_<$ vanishes along with $C_<$, i.e., when the axis is empty. Moreover, Eq. (5.16) shows that, when the C energy of the axis is positive, so is the quantity $D_<$, which means that the axis region is characterized by an angle deficit. This is in agreement with the corresponding results obtained from exact models of cosmic strings,[3-7] according to which a string with mass energy per unit length equal to μ produces an angle deficit $D=8\pi\mu$, with $0<\mu<\frac{1}{4}$. On the basis of these results and Eqs. (5.15) and (5.16) we are led to postulate the relation

$$|cp| = \exp(-C_<) = (1-4\mu)^{-1} \tag{5.17}$$

as a specifier of the physical meaning of the constants c and p appearing in our solution.

(2) For $\rho \gg |t|$,

$$C \approx \ln|cq| + \ln\left[1+\frac{1+5l^2}{2q^2\rho^2}\right] , \tag{5.18}$$

$$C_{,\eta} \approx \frac{1+l^2}{q^2\rho^3}\left[1+\frac{2t}{\rho}\right] , \tag{5.19a}$$

$$C_{,\zeta} \approx \frac{1+l^2}{q^2\rho^3}\left[1-\frac{2t}{\rho}\right] . \tag{5.19b}$$

These expressions imply that at spacetime infinity ($\rho \to \infty$) the solution under consideration is nonradiating. Instead, the C energy tends to the constant value $C_>$, where

$$C_> = \ln|cp| . \tag{5.20}$$

Combining Eqs. (4.4), (5.8), (5.15), and (5.20) we find that

$$D_> - D_< = \exp(-C_<) - \exp(-C_>) > 0 . \tag{5.21}$$

The interpretation of this relation is a direct result of the preceding analysis. Specifically, Eq. (5.21) expresses the fact that, from the standpoint of spacelike infinity, our solution represents a locally flat spacetime with a nonzero angle deficit. This deficit differs from the one obtained near the axis when the latter is occupied by a cosmic string by an amount which is determined by the effective mass of the gravitational field in the region $0 \le \rho < \infty$.

(3) *For* $\rho = |t| \to \infty$,

$$2C \approx \ln(c^2/2)\left[q^2+p^2-\frac{1+l^2}{\rho}\right] , \tag{5.22}$$

while

$$C_{,\eta} \approx \frac{1+l^2}{q^2+p^2}\left[1+\frac{1+l^2}{2(q^2+p^2)}\frac{1}{\rho}\right] , \tag{5.23a}$$

$$C_{,\zeta} \approx \frac{1+l^2}{4(q^2+p^2)}\frac{1}{\rho^2} \tag{5.23b}$$

when $t = \rho$, and

$$C_{,\eta} \approx -\frac{1+l^2}{4(q^2+p^2)}\frac{1}{\rho^2} , \tag{5.24a}$$

$$C_{,\zeta} \approx \frac{1+l^2}{q^2+p^2}\left[1+\frac{1+l^2}{2(q^2+p^2)}\frac{1}{\rho}\right] \tag{5.24b}$$

when $t = -\rho$.

Therefore, at past null infinity there is incoming radiation at a rate $(1+l^2)/(q^2+p^2)$, while at future null infinity there is outgoing radiation at the same rate.

This result, combined with the behavior of the C energy in the other regions of spacetime examined earlier, shows that our solution represents a cylindrical gravitational wave which converges towards the axis along the null direction $t = -\rho$, undergoes reflection at $t = 0$, and propagates out to infinity along $t = \rho$.

When $|cp| = 1$, the gravitational wave described by Eqs. (3.11), (3.12), and (4.5) propagates in empty spacetime. When $|cp| > 1$, on the other hand, the wave propagates in the background of a cosmic string upon which it converges at $t = 0$.

Cylindrical gravitational waves have, in general, two dynamical degrees of freedom which are referred to as the + and × radiation mode, respectively. As shown independently by Jordan, Ehlers, and Kundt,[23] and Kompaneets[24] the corresponding line element has the form given by Eq. (3.12) with (Y/X) representing the + mode and ω the × one. Thus, the Einstein-Rosen waves, for which the line element is given by (3.12) with $\omega = 0$, are cylindrical waves of the pure + type.

The Einstein vacuum equations for the line element (3.12), when written in terms of the functions (Y/X) and ω, show clearly that there is a feedback between the ingoing and outgoing waves of the same polarization as well as a rotation of the polarization vector when the × mode is present. These effects have been brought out clearly in the numerical integration of the field equations published by Piran, Safir, and Stark recently.[25]

In our case, the reflection of ingoing to outgoing waves, and vice versa, is expressed most clearly by Eqs. (5.23) and (5.24). Actually, these equations also express the rotation effect which accompanies the propagation of the wave pulse. However, instead of analyzing these aspects of our solution in greater detail, we prefer to turn to the discussion of another important issue regarding the interpretation of the above solution.

To begin with, let us note that Eqs. (5.22)–(5.24) retain their form when the parameter l vanishes. Since in this limit our solution reduces to the one found by Xanthopoulos recently it follows that the physical interpretation of the former holds for the latter as well. In particular, the Xanthopoulos metric is a radiating one, contrary to the conclusion of its author regarding this point.

Moreover, Xanthopoulos interpreted his solution as a rotating cosmic string when $|cp| > 1$. He based this interpretation on the fact that the Killing vector ∂_z is not hypersurface orthogonal as a result of the nonvanishing of ω. Here we adopt the above interpretation with the following qualification. Equations (5.1a) and (5.2) show that at timelike infinity the axis region is either flat and regular ($|cp| = 1$), or flat with a conical singularity ($|cp| > 1$). The former case obtains when the axis if empty, the latter when the axis is occupied by a static cosmic string. Therefore, the interpretation of a rotating string cannot be based on what is actually occurring on the axis.

We believe that the rotating string interpretation can be maintained as an expression of the effective action of the gravitational field plus nonrotating string, on the axis (when the latter is present) at spacelike infinity. Consider, to this effect, the last term in the line element (5.5). This term was transformed away following Eq. (5.5) in order to show the locally flat character of our spacetime model at spacelike infinity. Transformation (5.6), however, is not a proper one due to the periodic character of the angular coordinate ϕ. This means that the last term in Eq. (5.5) cannot be gauged away except locally. It will give rise to global effects analogous to the ones that arise in connection with stationary spacetimes where it is the timelike Killing vector which is not hypersurface orthogonal. We intend to present specific examples elsewhere.

C. The Weyl tensor

According to Eq. (3.12), the one-forms

$$\epsilon^0 \equiv f^{1/2} \mathbf{d}t, \quad \epsilon^1 \equiv (Y/X)^{1/2} (\mathbf{d}z - \omega\, \mathbf{d}\phi) , \quad \epsilon^2 \equiv (X/Y)^{1/2} \rho\, \mathbf{d}\phi, \quad \epsilon^3 \equiv f^{1/2} \mathbf{d}\rho \tag{5.25}$$

form an orthonormal set. Let $(\mathbf{e}_\mu)$, $\mu = 0,1,2,3$, be the set of vectors dual to (ϵ^μ). Then, the tetrad $(\mathbf{k}, \boldsymbol{l}, \mathbf{m}, \overline{\mathbf{m}})$, where

$$\mathbf{k} \equiv (\mathbf{e}_0 + \mathbf{e}_3)/\sqrt{2}, \quad \boldsymbol{l} \equiv (\mathbf{e}_0 - \mathbf{e}_3)/\sqrt{2} , \quad \mathbf{m} = (\mathbf{e}_1 + i\mathbf{e}_2)/\sqrt{2} \tag{5.26}$$

is a null one.

As shown by Chandrasekhar and Xanthopoulos,[26] the nonvanishing Weyl scalars in the above null tetrad are given by

$$\psi_0 = \frac{\overline{Z}+1}{2f(Z+\overline{Z})(Z+1)} \left[Z_{,\zeta} \left[\ln \frac{\rho}{f} \right]_{,\zeta} + Z_{,\zeta\zeta} - 2\frac{(Z_{,\zeta})^2}{Z+\overline{Z}} \right] , \tag{5.27a}$$

$$\psi_4 = \frac{\overline{Z}+1}{2f(Z+\overline{Z})(Z+1)} \left[Z_{,\eta} \left[\ln \frac{\rho}{f} \right]_{,\eta} + Z_{,\eta\eta} - 2\frac{(Z_{,\eta})^2}{Z+\overline{Z}} \right] , \tag{5.27b}$$

$$\psi_2 = \frac{1}{2f} \left[\frac{(\ln\rho)_{,\eta} (\ln\rho)_{,\zeta}}{4} - \frac{Z_{,\eta} \overline{Z}_{,\zeta}}{(Z+\overline{Z})^2} \right] , \tag{5.27c}$$

where

$$Z \equiv \rho(X/Y) + i\omega . \tag{5.27d}$$

Using Eq. (5.27), we find the following asymptotic behavior of the Weyl scalars.

For $\rho \gg \pm t > 0$,

$$\psi_0 \approx \psi_4 \approx \frac{-3(\pm l + i)}{2c^2 q^3} \frac{1}{\rho^3} , \tag{5.28a}$$

$$\psi_2 \approx -\frac{\pm l + i}{2c^2 q^3} \frac{1}{\rho^3} . \tag{5.28b}$$

These expressions show clearly that in the region $\rho \gg |t|$ the solution is nonradiating, confirming the corresponding result which was obtained using C-energy analysis.

For $t = \rho \to \infty$,

$$\psi_0 \approx -\tfrac{3}{4}(K + iL)\rho^{-5/2} , \tag{5.29a}$$

$$\psi_2 \approx -\tfrac{1}{4}(S + iR)\rho^{-3/2} , \tag{5.29b}$$

$$\psi_4 \approx -3\{K + (1+l^2)S - i[L - (1+l^2)R]\}\rho^{-1/2} , \tag{5.29c}$$

$$K \equiv \frac{lq + p}{c^2(p^2+q^2)^2}, \quad L \equiv \frac{lp - q}{c^2(p^2+q^2)^2} , \quad R \equiv [2(1+l^2)L + 4pqK]/(p^2+q^2)^2 , \quad S \equiv [4pqL - 2(1+l^2)K]/(p^2+q^2)^2 . \tag{5.29d}$$

The expressions for the corresponding measures read

$$|\psi_0| = \tfrac{3}{4} P \rho^{-5/2} , \tag{5.30a}$$

$$|\psi_2| = \tfrac{1}{2} P \rho^{-3/2} , \tag{5.30b}$$

$$|\psi_4| = 3 P \rho^{-1/2} , \tag{5.30c}$$

$$P \equiv (1+l^2)^{1/2}/c^2(p^2+q^2)^{3/2} . \tag{5.30d}$$

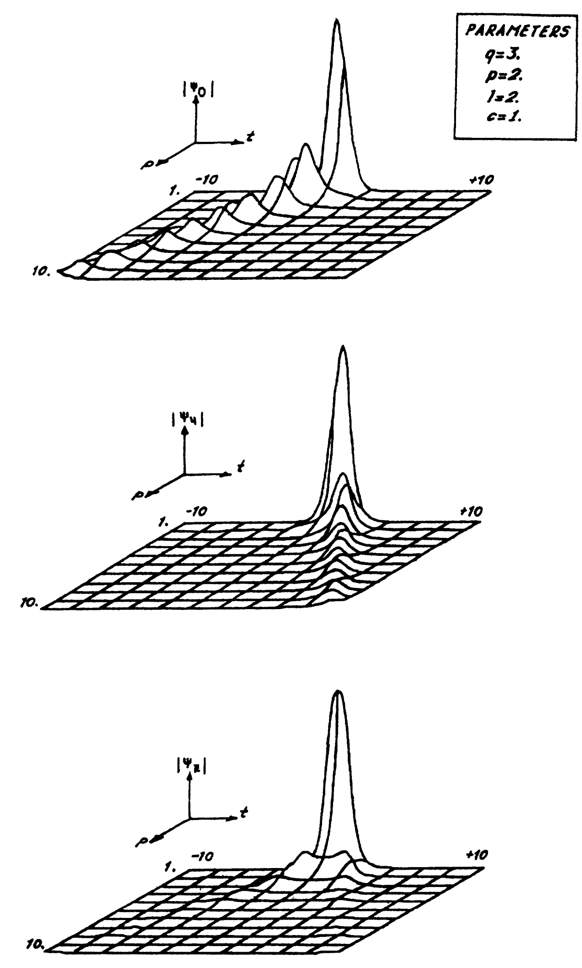

FIG. 1. The solitonlike behavior of the wave described in the text is brought out clearly in this series of graphs of the measures of the Weyl scalars ψ_0, ψ_4, and ψ_2 as functions of t and ρ. ψ_0 (ψ_4) represents a transverse gravitational wave in the $t = -\rho$ (ρ) direction, while ψ_2 gives the "Coulomb component" of the gravitational field. The values of the parameters q, p, l, and c are given in the square frame next to the first graph and the scale is relative. The maximum value of $|\psi_0|$ and $|\psi_4|$ is 1.12 while that of $|\psi_2|$ is 0.16.

Noting that ψ_0 (ψ_4) corresponds to a transverse wave in the l ($\mathbf{k}$) direction, while ψ_2 gives the Coulomb component of the field, we conclude from Eqs. (5.29) and (5.30) that in the region $t \approx \rho \rightarrow \infty$ our solution represents an outgoing wave.

In a similar fashion we find that for $-t \approx \rho - \infty$ the values of $|\psi_0|$ and $|\psi_4|$ are reversed while $|\psi_2|$ remains as in (5.30). Therefore, in this region we have an ingoing wave.

The asymptotic behavior of the Weyl scalars described above is brought out clearly in the computer diagrams of Fig. 1.

Computer calculations have also shown that the relation

$$9\psi_2^2 = \psi_0 \psi_4 , \tag{5.31}$$

which is an immediate consequence of Eqs. (5.28) and (5.30), holds everywhere and not only in the asymptotic regions considered above. Therefore, our solution is of Petrov type D.

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